

$$E(e_1, e_2)$$

DAVID BENSON AND ROBERT R. BRUNER

ABSTRACT. We give a counterexample to Theorem 5 in §18.2 of Margolis' book, "Spectra and the Steenrod Algebra," and make remarks about the proofs of some later theorems in the book that depend on it. The counterexample is a module which does not split as a sum of lightning flash modules and free modules.

## 1. INTRODUCTION

Let  $k$  be a field and  $E(e_1, e_2)$  be a graded exterior algebra on generators  $e_1$  and  $e_2$  with degrees satisfying  $0 < |e_1| < |e_2|$ . Theorem 5 in §18.2 of Margolis [2] states that every graded  $E(e_1, e_2)$ -module is a coproduct of free modules and lightning flashes. In this note, we give a simple counterexample to this statement.

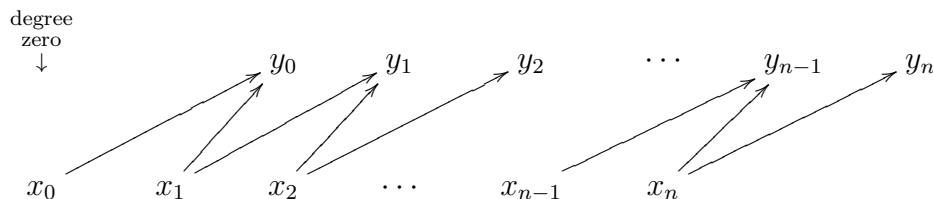
Statement (c) following Proposition 7 of the same section is true, but not because of Theorem 5. The proof of Theorem 8 in §18.3 depends on this statement. The proofs of Proposition 9 and Lemma 10 of the same section also depend on Theorem 5, and are used in Chapter 20. Fortunately, the paper of Adams and Margolis [1] provides correct proofs of these statements that do not rely on Theorem 5.

## 2. THE COUNTEREXAMPLE

In this section we display a bounded below module  $M$  for  $E(e_1, e_2)$  which is not isomorphic to a coproduct of free modules and lightning flashes.

First we note that every module for  $E(e_1, e_2)$  can be written as a direct sum of a free module and a module on which  $e_1 e_2$  acts as zero. So we may as well work with modules for  $E(e_1, e_2)/(e_1 e_2)$ .

We use the notation of §18.2 of Margolis. Let  $M(n)$  be the lightning module  $L(n, 0, 1)$  of dimension  $2n$ . Here is a picture of  $M(n)$ :



The shorter arrows represent the action of  $e_1$ , and the longer ones  $e_2$ . Thus a presentation of the module is given by  $e_1 x_{i+1} = e_2 x_i = y_i$  ( $0 \leq i \leq n-1$ ),  $e_1 x_0 = 0$ ,  $e_2 x_n = y_n$ . We arrange that the element  $x_0$  in  $M(n)$  is in degree zero, so that  $x_i$  has degree  $i(|e_2| - |e_1|)$  and  $y_i$  has

This work was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach.

degree  $|x_i| + |e_2|$ . Similarly,  $L(\infty, 0)$  is the infinite lightning flash obtained by letting this diagram continue to the right indefinitely.

Our counterexample is the module

$$M = \prod_{n=0}^{\infty} M(n).$$

To see that it is a counterexample, first note that  $e_1 M(n)$  is the linear span of  $y_0, \dots, y_{n-1}$ , so  $e_2^{-1} e_1 M(n)$  is the linear span of all the basis elements except  $x_n$ . Here, if  $U$  is a linear subspace of a module, we write  $e_2^{-1} U$  for the linear subspace consisting of the vectors whose image under  $e_2$  is in  $U$ .

Inductively, we see that for  $j > 0$ ,  $(e_2^{-1} e_1)^j M(n)$  is the linear span of the basis elements  $y_0, \dots, y_n, x_0, \dots, x_{n-j}$ . Thus  $x_0$  is in  $(e_2^{-1} e_1)^j M(n)$  if and only if  $j \leq n$ .

Taking degree zero parts, we have

$$((e_2^{-1} e_1)^j M)_0 = \prod_{j=n}^{\infty} M(n)_0.$$

Thus

$$(2.1) \quad \bigcap_{j \geq 0} ((e_2^{-1} e_1)^j M)_0 = 0,$$

and

$$((e_2^{-1} e_1)^j M)_0 / (e_2^{-1} e_1)^{j-1} M)_0$$

is one dimensional. On the other hand,  $x_0$  is in  $(e_2^{-1} e_1)^j L(\infty, 0)$  for all  $j > 0$ , so

$$\bigcap_{j \geq 0} ((e_2^{-1} e_1)^j L(\infty, 0))_0 \neq 0.$$

Since a finite sum is always a direct summand of the product, it follows that  $M$  has exactly one copy of each  $M(n)$  as a summand, and no summand isomorphic to  $L(\infty, 0)$ . Since  $e_1 M_0 = 0$ , no summand of the form  $L(\infty, 1)$ ,  $L(n, 1, 0)$  or  $L(n, 1, 1)$  can contribute to  $M_0$ ; and finally (2.1) shows that no summand of the form  $L(n, 0, 0)$  can contribute to  $M_0$ , since that intersection is non-zero for such a module. The summands we have identified do not exhaust  $M_0$ , and hence  $M$  cannot be a direct sum of lightning flash modules.

On the other hand, modules of finite type for  $E(e_1, e_2)$  can be shown to be direct sums of lightning flashes, by the method of filtrations of the forgetful functor to graded vector spaces. The proof is similar to but easier than the functorial filtration proof given in Ringel [3].

## REFERENCES

1. J. F. Adams and H. R. Margolis, *Modules over the Steenrod algebra*, Topology **10** (1971), 271–282.
2. H. R. Margolis, *Spectra and the Steenrod algebra*, North Holland, Amsterdam, 1983.
3. C. M. Ringel, *The indecomposable representations of the dihedral 2-groups*, Math. Ann. **214** (1975), 19–34.